

The Energy-Momentum tensor on low dimensional Spin^c manifolds

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On a compact surface endowed with any Spin^c structure, we give a formula involving the Energy-Momentum tensor in terms of geometric quantities. A new proof of a Bär-type inequality for the eigenvalues of the Dirac operator is given. The round sphere \mathbb{S}^2 with its canonical Spin^c structure satisfies the limiting case. Finally, we give a spinorial characterization of immersed surfaces in $\mathbb{S}^2 \times \mathbb{R}$ by solutions of the generalized Killing spinor equation associated with the induced Spin^c structure on $\mathbb{S}^2 \times \mathbb{R}$.

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1 Introduction

On a compact Spin surface, Th. Friedrich and E.C. Kim proved that any eigenvalue λ of the Dirac operator satisfies the equality [9, Thm. 4.5]:

$$\lambda^2 = \frac{\pi\chi(M)}{\text{Area}(M)} + \frac{1}{\text{Area}(M)} \int_M |T^\psi|^2 v_g, \quad (1.1)$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M and T^ψ is the field of quadratic forms called the Energy-Momentum tensor. It is given on the complement set of zeroes of the eigenspinor ψ by

$$T^\psi(X, Y) = g(\ell^\psi(X), Y) = \frac{1}{2} \text{Re} (X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2}),$$

for every $X, Y \in \Gamma(TM)$. Here ℓ^ψ is the field of symmetric endomorphisms associated with the field of quadratic forms T^ψ . We should point out that

since ψ is an eigenspinor, the zero set is discret [3]. The proof of Equality (1.1) relies mainly on a local expression of the covariant derivative of ψ and the use of the Schrödinger-Lichnerowicz formula. This equality has many direct consequences. First, since the trace of ℓ^ψ is equal to λ , we have by the Cauchy-Schwarz inequality that $|\ell^\psi|^2 \geq \frac{(tr(\ell^\psi))^2}{n} = \frac{\lambda^2}{2}$, where tr denotes the trace of ℓ^ψ . Hence, Equality (1.1) implies the Bär inequality [2] given by

$$\lambda^2 \geq \lambda_1^2 := \frac{2\pi\chi(M)}{Area(M)}. \quad (1.2)$$

Moreover, from Equality (1.1), Th. Friedrich and E.C. Kim [9] deduced that $\int_M \det(T^\psi)v_g = \pi\chi(M)$, which gives an information on the Energy-Momentum tensor without knowing the eigenspinor nor the eigenvalue. Finally, for any closed surface M in \mathbb{R}^3 of constant mean curvature H , the restriction to M of a parallel spinor on \mathbb{R}^3 is a generalized Killing spinor of eigenvalue $-H$ with Energy-Momentum tensor equal to the Weingarten tensor II (up to the factor $-\frac{1}{2}$) [21] and we have by (1.1)

$$H^2 = \frac{\pi\chi(M)}{Area(M)} + \frac{1}{4Area(M)} \int_M |II|^2 v_g.$$

Indeed, given any surface M carrying such a spinor field, Th. Friedrich [8] showed that the Energy-Momentum tensor associated with this spinor satisfies the Gauss-Codazzi equations and hence M is locally immersed into \mathbb{R}^3 .

Having a Spin^c structure on manifolds is a weaker condition than having a Spin structure because every Spin manifold has a trivial Spin^c structure. Additionally, any compact surface or any product of a compact surface with \mathbb{R} has a Spin^c structure carrying particular spinors. In the same spirit as in [14], when using a suitable conformal change, the second author [23] established a Bär-type inequality for the eigenvalues of the Dirac operator on a compact surface endowed with any Spin^c structure. In fact, any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geq \lambda_1^2 := \frac{2\pi\chi(M)}{Area(M)} - \frac{1}{Area(M)} \int_M |\Omega|^2 v_g, \quad (1.3)$$

where $i\Omega$ is the curvature form of the connection on the line bundle given by the Spin^c structure. Equality is achieved if and only if the eigenspinor ψ associated with the first eigenvalue λ_1 is a Killing Spin^c spinor, i.e., for every $X \in \Gamma(TM)$ the eigenspinor ψ satisfies

$$\begin{cases} \nabla_X \psi = -\frac{\lambda_1}{2} X \cdot \psi, \\ \Omega \cdot \psi = i|\Omega|\psi. \end{cases} \quad (1.4)$$

Here $X \cdot \psi$ denotes the Clifford multiplication and ∇ the spinorial Levi-Civita connection [7].

Studying the Energy-Momentum tensor on a compact Riemannian Spin or Spin^c manifolds has been done by many authors, since it is related to several geometric situations. Indeed, on compact Spin manifolds, J.P. Bourguignon and P. Gauduchon [5] proved that the Energy-Momentum tensor appears naturally

in the study of the variations of the spectrum of the Dirac operator. Th. Friedrich and E.C. Kim [10] obtained the Einstein-Dirac equation as the Euler-Lagrange equation of a certain functional. The second author extended these last two results to Spin^c manifolds [24]. Even if it is not a computable geometric invariant, the Energy-Momentum tensor is, up to a constant, the second fundamental form of an isometric immersion into a Spin or Spin^c manifold carrying a parallel spinor [21, 24]. For a better understanding of the tensor q^φ associated with a spinor field φ , the first author [12] studied Riemannian flows and proved that, if the normal bundle carries a parallel spinor ψ , the tensor q^φ associated with φ (the restriction of ψ to the flow) is the O'Neill tensor of the flow.

In this paper, we give a formula corresponding to (1.1) for any eigenspinor ψ of the square of the Dirac operator on compact surfaces endowed with any Spin^c structure (see Theorem 3.1). It is motivated by the following two facts: First, when we consider eigenvalues of the square of the Dirac operator, another tensor field is of interest. It is the skew-symmetric tensor field Q^ψ given by

$$Q^\psi(X, Y) = g(q^\psi(X), Y) = \frac{1}{2} \text{Re} (X \cdot \nabla_Y \psi - Y \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2}),$$

for all vector fields $X, Y \in \Gamma(TM)$. This tensor was studied by the first author in the context of Riemannian flows [12]. Second, we consider any compact surface M immersed in $\mathbb{S}^2 \times \mathbb{R}$ where \mathbb{S}^2 is the round sphere equipped with a metric of curvature one. The Spin^c structure on $\mathbb{S}^2 \times \mathbb{R}$, induced from the canonical one on \mathbb{S}^2 and the Spin structure on \mathbb{R} , admits a parallel spinor [22]. The restriction to M of this Spin^c structure is still a Spin^c structure with a generalized Killing spinor [24].

In Section 2, we recall some basic facts on Spin^c structures and the restrictions of these structures to hypersurfaces. In Section 3 and after giving a formula corresponding to (1.1) for any eigenspinor ψ of the square of the Dirac operator, we deduce a formula for the integral of the determinant of $T^\psi + Q^\psi$ and we establish a new proof of the Bär-type inequality (1.3). In Section 4, we consider the 3-dimensional case and treat examples of hypersurfaces in \mathbb{CP}^2 . In the last section, we come back to the question of a spinorial characterisation of surfaces in $\mathbb{S}^2 \times \mathbb{R}$. Here we use a different approach than the one in [25]. In fact, we prove that given any surface M carrying a generalized Killing spinor associated with a particular Spin^c structure, the Energy-Momentum tensor satisfies the four compatibility equations stated by B. Daniel [6]. Thus there exists a local immersion of M into $\mathbb{S}^2 \times \mathbb{R}$.

2 Preliminaries

In this section, we begin with some preliminaries concerning Spin^c structures and the Dirac operator. Details can be found in [18], [20], [7], [23] and [24].

The Dirac operator on Spin^c manifolds: Let (M^n, g) be a Riemannian manifold of dimension $n \geq 2$ without boundary. We denote by SOM

the SO_n -principal bundle over M of positively oriented orthonormal frames. A Spin^c structure of M is a Spin_n^c -principal bundle $(\mathrm{Spin}^c M, \pi, M)$ and an \mathbb{S}^1 -principal bundle $(\mathbb{S}^1 M, \pi, M)$ together with a double covering given by $\theta : \mathrm{Spin}^c M \rightarrow \mathrm{SOM} \times_M \mathbb{S}^1 M$ such that $\theta(ua) = \theta(u)\xi(a)$, for every $u \in \mathrm{Spin}^c M$ and $a \in \mathrm{Spin}_n^c$, where ξ is the 2-fold covering of Spin_n^c over $\mathrm{SO}_n \times \mathbb{S}^1$. Let $\Sigma M := \mathrm{Spin}^c M \times_{\rho_n} \Sigma_n$ be the associated spinor bundle where $\Sigma_n = \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$ and $\rho_n : \mathrm{Spin}_n^c \rightarrow \mathrm{End}(\Sigma_n)$ denotes the complex spinor representation. A section of ΣM will be called a spinor field. The spinor bundle ΣM is equipped with a natural Hermitian scalar product denoted by (\cdot, \cdot) . We define an L^2 -scalar product $\langle \psi, \varphi \rangle = \int_M (\psi, \varphi) v_g$, for any spinors ψ and φ . Additionally, any connection 1-form $A : T(\mathbb{S}^1 M) \rightarrow i\mathbb{R}$ on $\mathbb{S}^1 M$ and the connection 1-form ω^M on SOM , induce a connection on the principal bundle $\mathrm{SOM} \times_M \mathbb{S}^1 M$, and hence a covariant derivative ∇ on $\Gamma(\Sigma M)$ [7, 24]. The curvature of A is an imaginary valued 2-form denoted by $F_A = dA$, i.e., $F_A = i\Omega$, where Ω is a real valued 2-form on $\mathbb{S}^1 M$. We know that Ω can be viewed as a real valued 2-form on M [7, 17]. In this case $i\Omega$ is the curvature form of the associated line bundle L . It is the complex line bundle associated with the \mathbb{S}^1 -principal bundle via the standard representation of the unit circle. For every spinor ψ , the Dirac operator is locally defined by

$$D\psi = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi,$$

where (e_1, \dots, e_n) is a local oriented orthonormal tangent frame and “ \cdot ” denotes the Clifford multiplication. The Dirac operator is an elliptic, self-adjoint operator with respect to the L^2 -scalar product and verifies, for any spinor field ψ , the Schrödinger-Lichnerowicz formula

$$D^2\psi = \nabla^* \nabla \psi + \frac{1}{4} S\psi + \frac{i}{2} \Omega \cdot \psi \quad (2.1)$$

where $\Omega \cdot$ is the extension of the Clifford multiplication to differential forms given by $(e_i^* \wedge e_j^*) \cdot \psi = e_i \cdot e_j \cdot \psi$. For any spinor $\psi \in \Gamma(\Sigma M)$, we have [13]

$$(i\Omega \cdot \psi, \psi) \geq -\frac{c_n}{2} |\Omega|_g |\psi|^2, \quad (2.2)$$

where $|\Omega|_g$ is the norm of Ω , with respect to g given by $|\Omega|_g^2 = \sum_{i < j} (\Omega_{ij})^2$ in any orthonormal local frame and $c_n = 2[\frac{n}{2}]^{\frac{1}{2}}$. Moreover, equality holds in (2.2) if and only if $\Omega \cdot \psi = i\frac{c_n}{2} |\Omega|_g \psi$.

Every Spin manifold has a trivial Spin^c structure [7, 19]. In fact, we choose the trivial line bundle with the trivial connection whose curvature $i\Omega$ is zero. Also every Kähler manifold M of complex dimension m has a canonical Spin^c structure. Let \ltimes be the Kähler form defined by the complex structure J , i.e. $\ltimes(X, Y) = g(JX, Y)$ for all vector fields $X, Y \in \Gamma(TM)$. The complexified cotangent bundle

$$T^*M \otimes \mathbb{C} = \Lambda^{1,0} M \oplus \Lambda^{0,1} M$$

decomposes into the $\pm i$ -eigenbundles of the complex linear extension of the complex structure. Thus, the spinor bundle of the canonical Spin^c structure is given by

$$\Sigma M = \Lambda^{0,*} M = \oplus_{r=0}^m \Lambda^{0,r} M,$$

where $\Lambda^{0,r}M = \Lambda^r(\Lambda^{0,1}M)$ is the bundle of r -forms of type $(0,1)$. The line bundle of this canonical Spin^c structure is given by $L = (K_M)^{-1} = \Lambda^m(\Lambda^{0,1}M)$, where K_M is the canonical bundle of M [7, 19]. This line bundle L has a canonical holomorphic connection induced from the Levi-Civita connection whose curvature form is given by $i\Omega = -i\rho$, where ρ is the Ricci form given by $\rho(X, Y) = \text{Ric}(JX, Y)$. We point out that the canonical Spin^c structure on every Kähler manifold carries a parallel spinor [7, 22].

Spin^c hypersurfaces and the Gauss formula: Let \mathcal{Z} be an oriented $(n+1)$ -dimensional Riemannian Spin^c manifold and $M \subset \mathcal{Z}$ be an oriented hypersurface. The manifold M inherits a Spin^c structure induced from the one on \mathcal{Z} , and we have [24]

$$\Sigma M \simeq \begin{cases} \Sigma \mathcal{Z}|_M & \text{if } n \text{ is even,} \\ \Sigma^+ \mathcal{Z}|_M & \text{if } n \text{ is odd.} \end{cases}$$

Moreover Clifford multiplication by a vector field X , tangent to M , is given by

$$X \bullet \varphi = (X \cdot \nu \cdot \psi)|_M, \quad (2.3)$$

where $\psi \in \Gamma(\Sigma \mathcal{Z})$ (or $\psi \in \Gamma(\Sigma^+ \mathcal{Z})$ if n is odd), φ is the restriction of ψ to M , “ \cdot ” is the Clifford multiplication on \mathcal{Z} , “ \bullet ” that on M and ν is the unit normal vector. The connection 1-form defined on the restricted \mathbb{S}^1 -principal bundle $(P_{\mathbb{S}^1}M := P_{\mathbb{S}^1}\mathcal{Z}|_M, \pi, M)$, is given by $A = A^{\mathcal{Z}}|_M : T(P_{\mathbb{S}^1}M) = T(P_{\mathbb{S}^1}\mathcal{Z})|_M \longrightarrow i\mathbb{R}$. Then the curvature 2-form $i\Omega$ on the \mathbb{S}^1 -principal bundle $P_{\mathbb{S}^1}M$ is given by $i\Omega = i\Omega^{\mathcal{Z}}|_M$, which can be viewed as an imaginary 2-form on M and hence as the curvature form of the line bundle L^M , the restriction of the line bundle $L^{\mathcal{Z}}$ to M . For every $\psi \in \Gamma(\Sigma \mathcal{Z})$ ($\psi \in \Gamma(\Sigma^+ \mathcal{Z})$ if n is odd), the real 2-forms Ω and $\Omega^{\mathcal{Z}}$ are related by [24]

$$(\Omega^{\mathcal{Z}} \cdot \psi)|_M = \Omega \bullet \varphi - (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi. \quad (2.4)$$

We denote by $\nabla^{\Sigma \mathcal{Z}}$ the spinorial Levi-Civita connection on $\Sigma \mathcal{Z}$ and by ∇ that on ΣM . For all $X \in \Gamma(TM)$, we have the spinorial Gauss formula [24]:

$$(\nabla_X^{\Sigma \mathcal{Z}} \psi)|_M = \nabla_X \varphi + \frac{1}{2} II(X) \bullet \varphi, \quad (2.5)$$

where II denotes the Weingarten map of the hypersurface. Moreover, Let $D^{\mathcal{Z}}$ and D^M be the Dirac operators on \mathcal{Z} and M , after denoting by the same symbol any spinor and its restriction to M , we have

$$\nu \cdot D^{\mathcal{Z}} \varphi = \tilde{D} \varphi + \frac{n}{2} H \varphi - \nabla_{\nu}^{\Sigma \mathcal{Z}} \varphi, \quad (2.6)$$

where $H = \frac{1}{n} \text{tr}(II)$ denotes the mean curvature and $\tilde{D} = D^M$ if n is even and $\tilde{D} = D^M \oplus (-D^M)$ if n is odd.

3 The 2-dimensional case

In this section, we consider compact surfaces endowed with any Spin^c structure. We have

Theorem 3.1 *Let (M^2, g) be a Riemannian manifold and ψ an eigenspinor of the square of the Dirac operator D^2 with eigenvalue λ^2 associated with any Spin^c structure. Then we have*

$$\lambda^2 = \frac{S}{4} + |T^\psi|^2 + |Q^\psi|^2 + \Delta f + |Y|^2 - 2Y(f) + \left(\frac{i}{2}\Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right),$$

where f is the real-valued function defined by $f = \frac{1}{2}\ln|\psi|^2$ and Y is a vector field on TM given by $g(Y, Z) = \frac{1}{|\psi|^2}\text{Re}(D\psi, Z \cdot \psi)$ for any $Z \in \Gamma(TM)$.

Proof. Let $\{e_1, e_2\}$ be an orthonormal frame of TM . Since the spinor bundle ΣM is of real dimension 4, the set $\{\frac{\psi}{|\psi|}, \frac{e_1 \cdot \psi}{|\psi|}, \frac{e_2 \cdot \psi}{|\psi|}, \frac{e_1 \cdot e_2 \cdot \psi}{|\psi|}\}$ is orthonormal with respect to the real product $\text{Re}(\cdot, \cdot)$. The covariant derivative of ψ can be expressed in this frame as

$$\nabla_X \psi = \delta(X)\psi + \alpha(X) \cdot \psi + \beta(X)e_1 \cdot e_2 \cdot \psi, \quad (3.1)$$

for all vector fields X , where δ and β are 1-forms and α is a $(1, 1)$ -tensor field. Moreover β , δ and α are uniquely determined by the spinor ψ . In fact, taking the scalar product of (3.1) respectively with $\psi, e_1 \cdot \psi, e_2 \cdot \psi, e_1 \cdot e_2 \cdot \psi$, we get $\delta = \frac{d(|\psi|^2)}{2|\psi|^2}$ and

$$\alpha(X) = -\ell^\psi(X) + q^\psi(X) \quad \text{and} \quad \beta(X) = \frac{1}{|\psi|^2}\text{Re}(\nabla_X \psi, e_1 \cdot e_2 \cdot \psi).$$

Using (2.1), it follows that

$$\lambda^2 = \frac{\Delta(|\psi|^2)}{2|\psi|^2} + |\alpha|^2 + |\beta|^2 + |\delta|^2 + \frac{1}{4}S + \left(\frac{i}{2}\Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right).$$

Now it remains to compute the term $|\beta|^2$. We have

$$\begin{aligned} |\beta|^2 &= \frac{1}{|\psi|^4}\text{Re}(\nabla_{e_1} \psi, e_1 \cdot e_2 \cdot \psi)^2 + \frac{1}{|\psi|^4}\text{Re}(\nabla_{e_2} \psi, e_1 \cdot e_2 \cdot \psi)^2 \\ &= \frac{1}{|\psi|^4}\text{Re}(D\psi - e_2 \cdot \nabla_{e_2} \psi, e_2 \cdot \psi)^2 + \frac{1}{|\psi|^4}\text{Re}(D\psi - e_1 \cdot \nabla_{e_1} \psi, e_1 \cdot \psi)^2 \\ &= g(Y, e_1)^2 + g(Y, e_2)^2 + \frac{|d(|\psi|^2)|^2}{4|\psi|^4} - g(Y, \frac{d(|\psi|^2)}{|\psi|^2}) \\ &= |Y|^2 - 2Y(f) + \frac{|d(|\psi|^2)|^2}{4|\psi|^4}, \end{aligned}$$

which gives the result by using the fact that $\Delta f = \frac{\Delta(|\psi|^2)}{2|\psi|^2} + \frac{|d(|\psi|^2)|^2}{2|\psi|^4}$. \square

Remark 3.2 *Under the same conditions as Theorem 3.1, if ψ is an eigenspinor of D with eigenvalue λ , we get*

$$\lambda^2 = \frac{S}{4} + |T^\psi|^2 + \Delta f + \left(\frac{i}{2}\Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right).$$

In fact, in this case $Y = 0$ and

$$\begin{aligned} 0 &= \text{Re}(D\psi, e_1 \cdot e_2 \cdot \psi) = \text{Re}(e_1 \cdot \nabla_{e_1} \psi + e_2 \cdot \nabla_{e_2} \psi, e_1 \cdot e_2 \cdot \psi) \\ &= \text{Re}(-e_2 \cdot \nabla_{e_1} \psi + e_1 \cdot \nabla_{e_2} \psi, \psi) = 2Q^\psi(e_1, e_2)|\psi|^2. \end{aligned} \quad (3.2)$$

This was proven by Friedrich and Kim in [9] for a Spin structure on M .

In the following, we will give an estimate for the integral $\int_M \det(T^\psi + Q^\psi) v_g$ in terms of geometric quantities, which has the advantage that it does not depend on the eigenvalue λ nor on the eigenspinor ψ . This is a generalization of the result of Friedrich and Kim in [9] for Spin structures.

Theorem 3.3 *Let M be a compact surface and ψ any eigenspinor of D^2 associated with eigenvalue λ^2 . Then we have*

$$\int_M \det(T^\psi + Q^\psi) v_g \geq \frac{\pi \chi(M)}{2} - \frac{1}{4} \int_M |\Omega| v_g. \quad (3.3)$$

Equality in (3.3) holds if and only if either Ω is zero or has constant sign.

Proof. As in the previous theorem, the spinor $D\psi$ can be expressed in the orthonormal frame of the spinor bundle. Thus the norm of $D\psi$ is equal to

$$\begin{aligned} |D\psi|^2 &= \frac{1}{|\psi|^2} \operatorname{Re} (D\psi, \psi)^2 + \frac{1}{|\psi|^2} \sum_{i=1}^2 \operatorname{Re} (D\psi, e_i \cdot \psi)^2 + \frac{1}{|\psi|^2} \operatorname{Re} (D\psi, e_1 \cdot e_2 \cdot \psi)^2 \\ &= (\operatorname{tr} T^\psi)^2 |\psi|^2 + |Y|^2 |\psi|^2 + \frac{1}{|\psi|^2} \operatorname{Re} (D\psi, e_1 \cdot e_2 \cdot \psi)^2, \end{aligned} \quad (3.4)$$

where we recall that the trace of T^ψ is equal to $-\frac{1}{|\psi|^2} \operatorname{Re} (D\psi, \psi)$. On the other hand, by (3.2) we have that $\frac{1}{|\psi|^2} \operatorname{Re} (D\psi, e_1 \cdot e_2 \cdot \psi)^2 = 2|Q^\psi|^2 |\psi|^2$. Thus Equation (3.4) reduces to

$$\frac{|D\psi|^2}{|\psi|^2} = (\operatorname{tr} T^\psi)^2 + |Y|^2 + 2|Q^\psi|^2.$$

Now with the use of the equality $\operatorname{Re} (D^2\psi, \psi) = |D\psi|^2 - \operatorname{div} \xi$, where ξ is the vector field given by $\xi = |\psi|^2 Y$, we get

$$\lambda^2 + \frac{1}{|\psi|^2} \operatorname{div} \xi = (\operatorname{tr} T^\psi)^2 + |Y|^2 + 2|Q^\psi|^2. \quad (3.5)$$

An easy computation leads to $\frac{1}{|\psi|^2} \operatorname{div} \xi = \operatorname{div} Y + 2Y(f)$ where we recall that $f = \frac{1}{2} \ln(|\psi|^2)$. Hence substituting this formula into (3.5) and using Theorem 3.1 yields

$$\frac{S}{4} + \left(\frac{i}{2} \Omega \cdot \psi, \frac{\psi}{|\psi|^2} \right) + \Delta f + \operatorname{div} Y = (\operatorname{tr} T^\psi)^2 + |Q^\psi|^2 - |T^\psi|^2 = 2\det(T^\psi + Q^\psi).$$

Finally integrating over M and using the Gauss-Bonnet formula, we deduce the required result with the help of Equation (2.2). Equality holds if and only if $\Omega \cdot \psi = i|\Omega|\psi$. In the orthonormal frame $\{e_1, e_2\}$, the 2-form Ω can be written $\Omega = \Omega_{12} e_1 \wedge e_2$, where Ω_{12} is a function defined on M . Using the decomposition of ψ into positive and negative spinors ψ^+ and ψ^- , we find that the equality is attained if and only if

$$\Omega_{12} e_1 \cdot e_2 \cdot \psi^+ + \Omega_{12} e_1 \cdot e_2 \cdot \psi^- = i|\Omega_{12}| \psi^+ + i|\Omega_{12}| \psi^-,$$

which is equivalent to say that,

$$\Omega_{12} \psi^+ = -|\Omega_{12}| \psi^+ \quad \text{and} \quad \Omega_{12} \psi^- = |\Omega_{12}| \psi^-.$$

Now if $\psi^+ \neq 0$ and $\psi^- \neq 0$, we get $\Omega = 0$. Otherwise, it has constant sign. In the last case, we get that $\int_M |\Omega| v_g = 2\pi\chi(M)$, which means that the l.h.s. of this equality is a topological invariant. \square

Next, we will give another proof of the Bär-type inequality (1.3) for the eigenvalues of any Spin^c Dirac operator. The following theorem was proved by the second author in [23] using conformal deformation of the spinorial Levi-Civita connection.

Theorem 3.4 *Let M be a compact surface. For any Spin^c structure on M , any eigenvalue λ of the Dirac operator D to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \geq \frac{2\pi\chi(M)}{\text{Area}(M)} - \frac{1}{\text{Area}(M)} \int_M |\Omega| v_g. \quad (3.6)$$

Equality holds if and only if the eigenspinor ψ is a Spin^c Killing spinor, i.e., it satisfies $\Omega \cdot \psi = i|\Omega|\psi$ and $\nabla_X \psi = -\frac{\lambda}{2}X \cdot \psi$ for any $X \in \Gamma(TM)$.

Proof. With the help of Remark (3.2), we have that

$$\lambda^2 = \frac{S}{4} + |T^\psi|^2 + \Delta f + \left(\frac{i}{2}\Omega \cdot \psi, \frac{\psi}{|\psi|^2}\right). \quad (3.7)$$

Substituting the Cauchy-Schwarz inequality, i.e. $|T^\psi|^2 \geq \frac{\lambda^2}{2}$ and the estimate (2.2) into Equality (3.7), we easily deduce the result after integrating over M . Now the equality in (3.6) holds if and only if the eigenspinor ψ satisfies $\Omega \cdot \psi = i|\Omega|\psi$ and $|T^\psi|^2 = \frac{\lambda^2}{2}$. Thus, the second equality is equivalent to say that $\ell^\psi(X) = \frac{\lambda}{2}X$ for all $X \in \Gamma(TM)$. Finally, a straightforward computation of the spinorial curvature of the spinor field ψ gives in a local frame $\{e_1, e_2\}$ after using the fact $\beta = -(*\delta)$ that

$$\begin{aligned} \frac{1}{2}R_{1212} e_1 \cdot e_2 \cdot \psi &= \left(\frac{\lambda^2}{2} + e_1(\delta(e_1)) + e_2(\delta(e_2))\right)e_2 \cdot e_1 \cdot \psi - \lambda\delta(e_2)e_1 \cdot \psi \\ &\quad + \lambda\delta(e_1)e_2 \cdot \psi + \left(e_1(\delta(e_2)) - e_2(\delta(e_1))\right)\psi. \end{aligned}$$

Thus the scalar product with $e_1 \cdot \psi$ and $e_2 \cdot \psi$ implies that $\delta = 0$. Finally, $\beta = 0$ and the eigenspinor ψ is a Spin^c Killing spinor. \square

Now, we will give some examples where equality holds in (3.6) or in (3.3). Some applications of Theorem 3.1 are also given.

Examples:

1. Let \mathbb{S}^2 be the round sphere equipped with the standard metric of curvature one. As a Kähler manifold, we endow the sphere with the canonical Spin^c structure of curvature form equal to $i\Omega = -i\kappa$, where κ is the Kähler 2-form. Hence, we have $|\Omega| = |\kappa| = 1$. Furthermore, we mentioned that for the canonical Spin^c structure, the sphere carries parallel spinors, i.e., an eigenspinor associated with the eigenvalue 0 of the Dirac operator D . Thus equality holds in (3.6). On the other hand, the equality in (3.3) also holds, since the sign of the curvature form Ω is constant.

2. Let $f : M \rightarrow \mathbb{S}^3$ be an isometric immersion of a surface M^2 into the sphere equipped with its unique Spin structure and assume that the mean curvature H is constant. The restriction of a Killing spinor on \mathbb{S}^3 to the surface M defines a spinor field φ solution of the following equation [11]

$$\nabla_X \varphi = -\frac{1}{2}II(X) \bullet \varphi + \frac{1}{2}J(X) \bullet \varphi, \quad (3.8)$$

where II denotes the second fundamental form of the surface and J is the complex structure of M given by the rotation of angle $\frac{\pi}{2}$ on TM . It is easy to check that φ is an eigenspinor for D^2 associated with the eigenvalue $H^2 + 1$. Moreover $D\varphi = H\varphi + e_1 \cdot e_2 \cdot \varphi$, so that $Y = 0$. Moreover the tensor $T^\varphi = \frac{1}{2}II$ and $Q^\varphi = \frac{1}{2}J$. Hence by Theorem 3.1, and since the norm of φ is constant, we obtain

$$H^2 + \frac{1}{2} = \frac{1}{4}S + \frac{1}{4}|II|^2.$$

3. On two-dimensional manifolds, we can define another Dirac operator associated with the complex structure J given by $\tilde{D} = Je_1 \cdot \nabla_{e_1} + Je_2 \cdot \nabla_{e_2} = e_2 \cdot \nabla_{e_1} - e_1 \cdot \nabla_{e_2}$. Since \tilde{D} satisfies $D^2 = (\tilde{D})^2$, all the above results are also true for the eigenvalues of \tilde{D} .
4. Let M^2 be a surface immersed in $\mathbb{S}^2 \times \mathbb{R}$. The product of the canonical Spin^c structure on \mathbb{S}^2 and the unique Spin structure on \mathbb{R} define a Spin^c structure on $\mathbb{S}^2 \times \mathbb{R}$ carrying parallel spinors [22]. Moreover, by the Schrödinger-Lichnerowicz formula, any parallel spinor ψ satisfies $\Omega^{\mathbb{S}^2 \times \mathbb{R}} \cdot \psi = i\psi$, where $\Omega^{\mathbb{S}^2 \times \mathbb{R}}$ is the curvature form of the auxiliary line bundle. Let ν be a unit normal vector field of the surface. We then write $\partial t = T + f\nu$ where T is a vector field on TM with $\|T\|^2 + f^2 = 1$. On the other hand, the vector field T splits into $T = \nu_1 + h\partial t$ where ν_1 is a vector field on the sphere. The scalar product of the first equation by T and the second one by ∂t gives $\|T\|^2 = h$ which means that $h = 1 - f^2$. Hence the normal vector field ν can be written as $\nu = f\partial t - \frac{1}{f}\nu_1$. As we mentioned before, the Spin^c structure on $\mathbb{S}^2 \times \mathbb{R}$ induces a Spin^c structure on M with induced auxiliary line bundle. Next, we want to prove that the curvature form of the auxiliary line bundle of M is equal to $i\Omega(e_1, e_2) = -if$, where $\{e_1, e_2\}$ denotes a local orthonormal frame on TM . Since the spinor ψ is parallel, we have by [22] that for all $X \in T(\mathbb{S}^2 \times \mathbb{R})$ the equality $\text{Ric}^{\mathbb{S}^2 \times \mathbb{R}} X \cdot \psi = i(X \lrcorner \Omega^{\mathbb{S}^2 \times \mathbb{R}}) \cdot \psi$. Therefore, we compute

$$\begin{aligned} (\nu \lrcorner \Omega^{\mathbb{S}^2 \times \mathbb{R}}) \bullet \varphi &= (\nu \lrcorner \Omega^{\mathbb{S}^2 \times \mathbb{R}}) \cdot \nu \cdot \psi|_M = i\nu \cdot \text{Ric}^{\mathbb{S}^2 \times \mathbb{R}} \nu \cdot \psi|_M \\ &= -\frac{1}{f}i\nu \cdot \nu_1 \cdot \psi|_M = i\nu \cdot (\nu - f\partial t) \cdot \psi|_M \\ &= (-i\psi - if\nu \cdot \partial t \cdot \psi)|_M. \end{aligned}$$

Hence by Equation (2.4), we get that $\Omega \bullet \varphi = -i(f\nu \cdot \partial t \cdot \psi)|_M$. The scalar product of the last equality with $e_1 \cdot e_2 \cdot \psi$ gives

$$\Omega(e_1, e_2)|\varphi|^2 = -f\text{Re}(i\nu \cdot \partial t \cdot \psi, e_1 \cdot e_2 \cdot \psi)|_M = -f\text{Re}(i\partial t \cdot \psi, \psi)|_M.$$

We now compute the term $i\partial t \cdot \psi$. For this, let $\{e'_1, Je'_1\}$ be a local orthonormal frame of the sphere \mathbb{S}^2 . The complex volume form acts as the

identity on the spinor bundle of $\mathbb{S}^2 \times \mathbb{R}$, hence $\partial t \cdot \psi = e'_1 \cdot J e'_1 \cdot \psi$. But we have

$$\Omega^{\mathbb{S}^2 \times \mathbb{R}} \cdot \psi = -\rho \cdot \psi = -\kappa \cdot \psi = -e'_1 \cdot J e'_1 \cdot \psi.$$

Therefore, $i\partial t \cdot \psi = \psi$. Thus we get $\Omega(e_1, e_2) = -f$. Finally,

$$(i\Omega \bullet \varphi, \varphi) = f \operatorname{Re} (\nu \cdot \partial t \cdot \psi, \psi)|_M = -fg(\nu, \partial t)|\varphi|^2 = -f^2|\varphi|^2.$$

Hence Equality in Theorem 3.1 is just

$$H^2 = \frac{S}{4} + \frac{1}{4}|II|^2 - \frac{1}{2}f^2.$$

4 The 3-dimensional case

In this section, we will treat the 3-dimensional case.

Theorem 4.1 *Let (M^3, g) be an oriented Riemannian manifold. For any Spin^c structure on M , any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \leq \frac{1}{\operatorname{vol}(M, g)} \int_M (|T^\psi|^2 + \frac{S}{4} + \frac{|\Omega|}{2}) v_g.$$

Equality holds if and only if the norm of ψ is constant and $\Omega \cdot \psi = i|\Omega|\psi$.

Proof. As in the proof of Theorem 3.1, the set $\{\frac{\psi}{|\psi|}, \frac{e_1 \cdot \psi}{|\psi|}, \frac{e_2 \cdot \psi}{|\psi|}, \frac{e_3 \cdot \psi}{|\psi|}\}$ is orthonormal with respect to the real product $\operatorname{Re}(\cdot, \cdot)$. The covariant derivative of ψ can be expressed in this frame as

$$\nabla_X \psi = \eta(X)\psi + \ell(X) \cdot \psi, \quad (4.1)$$

for all vector fields X , where η is a 1-form and ℓ is a $(1, 1)$ -tensor field. Moreover $\eta = \frac{d(|\psi|^2)}{2|\psi|^2}$ and $\ell(X) = -\ell^\psi(X)$. Using (2.1), it follows that

$$\begin{aligned} \lambda^2 &= \frac{\Delta(|\psi|^2)}{2|\psi|^2} + |T^\psi|^2 + \frac{|d(|\psi|^2)|^2}{4|\psi|^4} + \frac{1}{4}S + (\frac{i}{2}\Omega \cdot \psi, \frac{\psi}{|\psi|^2}) \\ &= \Delta f - \frac{|d(|\psi|^2)|^2}{2|\psi|^4} + |T^\psi|^2 + \frac{1}{4}S + (\frac{i}{2}\Omega \cdot \psi, \frac{\psi}{|\psi|^2}). \end{aligned}$$

By the Cauchy-Schwarz inequality, we have $\frac{1}{2}(i\Omega \cdot \psi, \frac{\psi}{|\psi|^2}) \leq \frac{1}{2}|\Omega|$. Integrating over M and using the fact that $|d(|\psi|^2)|^2 \geq 0$, we get the result. \square

Example 4.2 *Let M^3 be a 3-dimensional Riemannian manifold immersed in \mathbb{CP}^2 with constant mean curvature H . Since \mathbb{CP}^2 is a Kähler manifold, we endow it with the canonical Spin^c structure whose line bundle has curvature equal to $-3i\kappa$. Moreover, by the Schrödinger-Lichnerowicz formula we have that any parallel spinor ψ satisfies $\Omega^{\mathbb{CP}^2} \cdot \psi = 6i\psi$. As in the previous example, we compute*

$$(\nu \lrcorner \Omega^{\mathbb{CP}^2}) \bullet \varphi = i(\nu \cdot \operatorname{Ric}^{\mathbb{CP}^2}(\nu) \cdot \psi)|_M = -3i\varphi.$$

Finally, $\Omega \bullet \varphi = 3i\varphi$. Using Equation (2.6), we have that $-\frac{3}{2}H$ is an eigenvalue of D . Since the norm of φ is constant, equality holds in Theorem 4.1 and hence

$$\frac{9}{4}H^2 + \frac{3}{2} = \frac{S}{4} + \frac{1}{4}|II|^2.$$

5 Characterization of surfaces in $\mathbb{S}^2 \times \mathbb{R}$

In this section, we characterize the surfaces in $\mathbb{S}^2 \times \mathbb{R}$ by solutions of the generalized Killing spinors equation which are restrictions of parallel spinors of the canonical Spin^c -structure on $\mathbb{S}^2 \times \mathbb{R}$ (see also [25] for a different proof). First recall the compatibility equations for characterization of surfaces in $\mathbb{S}^2 \times \mathbb{R}$ established by B. Daniel [6, Thm 3.3]:

Theorem 5.1 *Let (M, g) be a simply connected Riemannian manifold of dimension 2, $A : TM \rightarrow TM$ a field of symmetric operator and T a vector field on TM . We denote by f a real valued function such that $f^2 + ||T||^2 = 1$. Assume that M satisfies the Gauss-Codazzi equations, i.e. $G = \det A + f^2$ and*

$$d^\nabla A(X, Y) := (\nabla_X A)Y - (\nabla_Y A)X = f(g(Y, T)X - g(X, T)Y),$$

where G is the gaussian curvature, and the following equations

$$\nabla_X T = fA(X), \quad X(f) = -g(AX, T).$$

Then there exists an isometric immersion of M into $\mathbb{S}^2 \times \mathbb{R}$ such that the Weingarten operator is A and $\partial_t = T + f\nu$, where ν is the normal vector field to the surface M .

Now using this characterization theorem, we state our result:

Theorem 5.2 *Let M be an oriented simply connected Riemannian manifold of dimension 2. Let T be a vector field and denote by f a real valued function such that $f^2 + ||T||^2 = 1$. Denote by A a symmetric endomorphism field of TM . The following statements are equivalent:*

1. *There exists an isometric immersion of M into $\mathbb{S}^2 \times \mathbb{R}$ of Weingarten operator A such that $\partial_t = T + f\nu$, where ν is the unit normal vector field of the surface.*
2. *There exists a Spin^c structure on M whose line bundle has a connection of curvature given by $i\Omega = -if\lrcorner$, such that it carries a non-trivial solution φ of the generalized Killing spinor equation $\nabla_X \varphi = -\frac{1}{2}AX \bullet \varphi$, with $T \bullet \varphi = -f\varphi + \bar{\varphi}$.*

Proof. We begin with $1 \Rightarrow 2$. The existence of such a Spin^c structure is assured by the restriction of the canonical one on $\mathbb{S}^2 \times \mathbb{R}$. Moreover, using the spinorial Gauss formula (2.5), any parallel spinor ψ on $\mathbb{S}^2 \times \mathbb{R}$ induces a generalized Killing spinor $\varphi = \psi|_M$ with A the Weingarten map of the surface M . Hence it remains

to show the relation $T \bullet \varphi = -f\varphi + \bar{\varphi}$. In fact, using that $\Omega^{\mathbb{S}^2 \times \mathbb{R}} \cdot \psi = i\psi$, we write in the frame $\{e_1, e_2, \nu\}$

$$\Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, e_2)e_1 \cdot e_2 \cdot \psi + \Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, \nu)e_1 \cdot \nu \cdot \psi + \Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_2, \nu)e_2 \cdot \nu \cdot \psi = i\psi. \quad (5.1)$$

By the previous example in Section 3, we know that $\Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, e_2) = -f$. For the other terms, we compute

$$\Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, \nu) = \Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, \frac{1}{f}\partial t - \frac{1}{f}T) = -\frac{1}{f}g(T, e_2)\Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, e_2) = g(T, e_2),$$

where the term $\Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_1, \partial t)$ vanishes since we can split e_1 into a sum of vectors on the sphere and on \mathbb{R} . Similarly, we find that $\Omega^{\mathbb{S}^2 \times \mathbb{R}}(e_2, \nu) = -g(T, e_1)$. By substituting these values into (5.1) and taking Clifford multiplication with $e_1 \cdot e_2$, we get the desired property. For $2 \Rightarrow 1$, a straightforward computation for the spinorial curvature of the generalized Killing spinor φ yields on a local frame $\{e_1, e_2\}$ of TM that

$$(-G + \det A)e_1 \bullet e_2 \bullet \varphi = -(d^\nabla A)(e_1, e_2) \bullet \varphi + if\varphi. \quad (5.2)$$

In the following, we will prove that the spinor field $\theta := i\varphi - if\bar{\varphi} + JT \bullet \varphi$ is zero. For this, it is sufficient to prove that its norm vanishes. Indeed, we compute

$$|\theta|^2 = |\varphi|^2 + f^2|\varphi|^2 + \|T\|^2|\varphi|^2 - 2\operatorname{Re}(i\varphi, if\bar{\varphi}) + 2\operatorname{Re}(i\varphi, JT \bullet \varphi) \quad (5.3)$$

From the relation $T \bullet \varphi = -f\varphi + \bar{\varphi}$ we deduce that $\operatorname{Re}(\varphi, \bar{\varphi}) = f|\varphi|^2$ and the equalities

$$g(T, e_1)|\varphi|^2 = \operatorname{Re}(ie_2 \bullet \varphi, \varphi) \quad \text{and} \quad g(T, e_2)|\varphi|^2 = -\operatorname{Re}(ie_1 \bullet \varphi, \varphi).$$

Therefore, Equation (5.3) becomes

$$\begin{aligned} |\theta|^2 &= 2|\varphi|^2 - 2f^2|\varphi|^2 + 2\operatorname{Re}(i\varphi, JT \bullet \varphi) \\ &= 2|\varphi|^2 - 2f^2|\varphi|^2 + 2g(JT, e_1)\operatorname{Re}(i\varphi, e_1 \bullet \varphi) + 2g(JT, e_2)\operatorname{Re}(i\varphi, e_2 \bullet \varphi) \\ &= 2|\varphi|^2 - 2f^2|\varphi|^2 + 2g(JT, e_1)g(T, e_2)|\varphi|^2 - 2g(JT, e_2)g(T, e_1)|\varphi|^2 \\ &= 2|\varphi|^2 - 2f^2|\varphi|^2 - 2g(JT, e_1)^2|\varphi|^2 - 2g(T, e_1)^2|\varphi|^2 \\ &= 2|\varphi|^2 - 2f^2|\varphi|^2 - 2\|T\|^2|\varphi|^2 = 0. \end{aligned}$$

Thus, we deduce $if\varphi = -f^2e_1 \cdot e_2 \cdot \varphi - fJT \cdot \varphi$, where we use the fact that $\bar{\varphi} = ie_1 \bullet e_2 \bullet \varphi$. In this case, Equation (5.2) can be written as

$$(-G + \det A + f^2)e_1 \bullet e_2 \bullet \varphi = -((d^\nabla A)(e_1, e_2) + fJT) \bullet \varphi.$$

This is equivalent to say that both terms $R_{1212} + \det A + f^2$ and $(d^\nabla A)(e_1, e_2) + fJT$ are equal to zero. In fact, these are the Gauss-Codazzi equations in Theorem 5.1. In order to obtain the two other equations, we simply compute the derivative of $T \cdot \varphi = -f\varphi + \bar{\varphi}$ in the direction of X to get

$$\begin{aligned} \nabla_X T \bullet \varphi + T \bullet \nabla_X \varphi &= \nabla_X T \bullet \varphi - \frac{1}{2}T \bullet A(X) \bullet \varphi \\ &= -X(f)\varphi - f\nabla_X \varphi + \nabla_X \bar{\varphi} \\ &= -X(f)\varphi + \frac{1}{2}fAX \bullet \varphi + \frac{1}{2}AX \bullet \bar{\varphi} \\ &= -X(f)\varphi + \frac{1}{2}fAX \bullet \varphi + \frac{1}{2}AX \bullet (T \bullet \varphi + f\varphi). \end{aligned}$$

This reduces to $\nabla_X T \bullet \varphi + g(T, A(X))\varphi = -X(f)\varphi + fA(X) \bullet \varphi$. Hence we obtain $X(f) = -g(A(X), T)$ and $\nabla_X T = fA(X)$ which finishes the proof. \square

Remark 5.3 *The second condition in Theorem 5.2 is equivalent to the existence of a Spin^c structure whose line bundle L verifies $c_1(L) = [\frac{i}{2\pi}f\lrcorner]$ and $f\lrcorner$ is a closed 2-form. This Spin^c structure carries a non-trivial solution φ of the generalized Killing spinor equation $\nabla_X \varphi = -\frac{1}{2}AX \bullet \varphi$, with $T \bullet \varphi = -f\varphi + \bar{\varphi}$.*

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